The Nonsteady Flow of Fluids Through Expansible Tubes

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A digital computer solution of the equations of fluid motion predicts the pressures in an expansible tube through which liquid is flowing.

To obtain a stable solution, it is necessary to assume that the rate of change of area with time is proportional to the rate of change of velocity with time. This assumption is shown to be a reasonable one.

Calculations for the rate of travel of pressure pulses are in good agreement with experimental results.

Since the initial work of Thomas Young (*4*) in 1808 in the area of nonsteady fluid flow in elastic tubes, many investigators have become interested in this area of study. Young was interested in predicting the velocity of propagation of a pulse wave in elastic tubes as a function of the properties of the tube and the fluid. The impetus of his work was a desire to learn more about the movement of blood in human blood vessels. Most of the investigators since Young have been interested in continuing and improving the work started by Young on predicting the velocity of propagation of a pressure pulse in an elastic tube. Little attention has been given to the prediction of the amount of pressure damping which takes place when a pressure pulse moves along an elastic tube.

The growing interest in Bioengineering and the increase in communication between the medical and engineering disciplines have encouraged engineers to enter this field. Also, novel uses of expansible tubes by the space agency and industry have increased the need for a better understanding of the nonsteady flow of fluids through expansible tubes. In this article, a set of equations is presented which may be easily solved on a digital computer to predict the movement of a pressure pulse through an expansible tube. Both the change of shape of the pulse and the propagation velocity can be obtained from this solution which includes the effect of viscosity on the damping of the pressure pulse.

DEVELOPMENT OF EQUATIONS

The model used in this development is that of a pressure pulse superimposed upon the steady laminar flow of a viscous liquid through an expansible tube. The equations used are the equations of motion, the equation of continuity, and an equation of state of the tube which expresses the radius of the tube as a function of gage pressure.

The equations of motion are the Navier-Stokes equations subjected to the restrictions of axial symmetry, constant density, and constant viscosity, and written in cylindrical coordinates:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial P}{\partial z} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}\right) \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial z} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial P}{\partial r} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{r^2}\right) \quad (2)$$

The equation of continuity is

$$\frac{1}{r}\frac{\partial (rv)}{\partial r} + \frac{\partial u}{\partial z} = 0 \quad (3)$$

Neglecting the inertia of the tube walls, a force balance gives this equation of state of the tube:

$$P(R) = E\delta\left(\frac{1}{R_u} - \frac{1}{R}\right) \quad (4)$$

The simplification of Equation (3) and the left side of Equation (1) is similar to that used by Lambert (*1*) and proceeds by eliminating the radial coordinate by defining an average axial velocity $\overline{u}$ as

$$A\overline{u} = \int_0^R 2\pi r u dr \quad (5)$$

and an average pressure as

$$A\overline{P} = \int_0^R 2\pi r P dr \quad (6)$$

Five other relationships are needed to simplify these equations.

1. There are two sources of radial motion of a particle at the tube wall: the movement of the wall, and the radial component of the axial velocity due to the slope of the tube wall. These lead to the following kinematic relationship at the inner wall of the tube:

$$v(R) = \frac{\partial R}{\partial t} + u(R)\frac{\partial R}{\partial z} \quad (7)$$

By using Leibnitz's rule for differentiation under the

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integral sign and Equations (5) and (6) the following four expressions are obtained

$$\int_0^R 2\pi r \frac{\partial u}{\partial z} dr = \frac{\partial(\bar{u}A)}{\partial z} - u(R)\frac{\partial A}{\partial z} \quad (8)$$

$$\int_0^R 2\pi r \frac{\partial u}{\partial t} dr = \frac{\partial(\bar{u}A)}{\partial t} - u(R)\frac{\partial R}{\partial z} \quad (9)$$

$$\int_0^R 2\pi r \frac{\partial u^2}{\partial z} = \frac{\partial}{\partial z}\int_0^R 2\pi r u^2 dr - 2\pi R u^2(R)\frac{\partial R}{\partial z} \quad (10)$$

$$\int_0^R 2\pi r \frac{\partial P}{\partial z} dr = \frac{\partial(\bar{P}A)}{\partial z} - P(R)\frac{\partial A}{\partial z} \quad (11)$$

Simplification of the equation of continuity (3) is accomplished by multiplying each term by $2\pi r dr$ and integrating from 0 to R. Utilizing Equations (7), (8), and (9), this equation simplifies to

$$\frac{\partial A}{\partial t} + \frac{\partial(\bar{u}A)}{\partial z} = 0 \quad (12)$$

Simplification of the axial equation of motion (1) follows the same procedure as used for the equation of continuity, resulting in

$$\frac{\partial(\bar{u}A)}{\partial t} + \frac{\partial}{\partial z}\int_0^R 2\pi r u^2 dr + \frac{1}{\rho}\frac{\partial(\bar{P}A)}{\partial z} - \frac{P(R)}{\rho}\frac{\partial A}{\partial z} = \int_0^R 2\pi r(V.T.)dr \quad (13)$$

where (V.T.) stands for the viscosity terms in Equation (1).

To complete the simplification of the left side of Equation (13) it is necessary to make two assumptions:

1. Assume that the axial velocity profile is parabolic with the radius of the tube,

$$u = 2\bar{u}\left(1 - \frac{r^2}{R^2}\right) \quad (14)$$

2. Assume that

$$P(R) = \bar{P} \quad (15)$$

Using Equation (14), the second term of Equation (13) becomes,

$$\frac{\partial}{\partial z}\int_0^R 2\pi r u^2 dr = \frac{4}{3}\bar{u}^2 A \quad (16)$$

The justification for assumption one will be discovered in connection with the simplification of the viscosity term. The second assumption implies that the pressure at the wall of the tube can adequately be approximated by the average pressure at any cross section of the tube. The best verification of this assumption can be found by looking at results obtained for conditions of steady state flow by Sheppard (3). For the slope of the tube wall, $\partial R/\partial z$, which might be encountered when a pressure pulse is put into an expansible tube of the type used in this study, the pressure at the wall deviates from the average by less than 1 in 10^6. Since it is the slope of the tube wall which creates a pressure drop in the radial direction in both steady and nonsteady flow, the deviation between the pressure at the wall and the average pressure at some cross section should be of the same order of magnitude as in the steady state study. The driving force for secondary flow is $\partial P/\partial r$ and the effect of this flow is described in Equation (2). Since the driving force for secondary flow is very small in comparison to that for axial flow, Equation (2) will not be used in this solution. This does not completely eliminate the effect of radial flow since this is retained through the use of the equation of continuity, Equation (3). Using Equations (14) and (15), Equation (13) becomes

$$\frac{\partial(A\bar{u})}{\partial t} + \frac{4}{3}\frac{\partial(A\bar{u}^2)}{\partial z} + \frac{A}{\rho}\frac{\partial\bar{P}}{\partial z} = \int_0^R 2\pi r(\text{V.T.})dr \quad (17)$$

By using an order of magnitude study, Morgan (2) showed that of the viscous terms in the right side of Equation (1), $\partial^2 u/\partial z^2$, is much smaller than the other two terms. To evaluate the remaining two terms, consider the case where the length of the pulse wave in the tube is much greater than the diameter of the tube. In this case the pressure-length profile will look much like it does in the steady state flow case. The variation of axial velocity with the radius will be very nearly parabolic; this was found to be the case by Morgan (2) and Sheppard (3) even for maximum tube expansion. Substitution of Equation (14) for velocity in the viscous terms and integrating over the cross sectional area yields

$$\int_0^R 2\pi r\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)dr = -8\nu\pi\bar{u} \quad (18)$$

Expanding the partials of Equation (17) and using Equations (12) and (18) give the simplified form of the equation of motion,

$$\frac{\partial\bar{u}}{\partial t} + \frac{5}{3}\bar{u}\frac{\partial\bar{u}}{\partial z} + \frac{1}{3}\frac{\bar{u}^2}{A}\frac{\partial A}{\partial z} + \frac{1}{\rho}\frac{\partial\bar{P}}{\partial z} + \frac{8\pi\nu\bar{u}}{A} = 0 \quad (19)$$

Finally, by using the second assumption the equation of state of the tube becomes

$$\bar{P} = E\delta\left(\frac{1}{R_u} - \frac{1}{R}\right) \quad (20)$$

which yields the final equation necessary to begin the solution.

The three equations to be used in the solution are Equations (12), (19), and (20). The first task is to eliminate either A or $\bar{P}$ from Equations (12) and (19). This can be done by the partial differentiation of Equation (20), with respect to z and writing the radius, R, in terms of cross sectional area, A, as

$$\frac{\partial\bar{P}}{\partial z} = \frac{E\delta\sqrt{\pi}}{2A^{3/2}}\frac{\partial A}{\partial z} \quad (21)$$

Since the solution is easier to handle in terms of A than in terms of $\bar{P}$ and since there is no problem in numerically transforming values of A into values of $\bar{P}$ by way of Equation (20), pressure is eliminated from Equation (19). This yields the following set of equations with two dependent variables, A and $\bar{u}$. and two independent variables, z and t:

$$\frac{\partial\bar{u}}{\partial t} + \frac{5}{3}\bar{u}\frac{\partial\bar{u}}{\partial z} + \left(\frac{\bar{u}^2}{3A} + \frac{C_1}{A^{3/2}}\right)\frac{\partial A}{\partial z} + \frac{C_2\bar{u}}{A} = 0 \quad (22)$$

$$\frac{\partial A}{\partial t} + \bar{u}\frac{\partial A}{\partial z} + A\frac{\partial\bar{u}}{\partial z} = 0 \quad (23)$$

where

$$C_1 = \frac{E\delta\sqrt{\pi}}{2\rho}; \quad \text{and} \quad C_2 = 8\pi\nu$$

One approach to a solution is to write these equations as differential equations in t and difference equations in z as

$$\frac{d\bar{u}}{dt} + \frac{5}{3}\bar{u}\frac{\Delta\bar{u}}{\Delta z} + \left(\frac{\bar{u}^2}{3A} + \frac{C_1}{A^{3/2}}\right)\frac{\Delta A}{\Delta z} + \frac{C_2\bar{u}}{A} = 0 \quad (24)$$

and

$$\frac{dA}{dt} + \bar{u}\frac{\Delta A}{\Delta z} + A\frac{\Delta\bar{u}}{\Delta z} = 0 \quad (25)$$

The initial conditions for the numerical integration of these equations are the area and velocity profile as a function of time at the inlet of the expansible tube and the area and velocity at time zero at evenly spaced intervals along the axis of the tube. The straightforward integration on the time variable at discrete intervals along z becomes numerically unstable, unless for a given area-time profile at the inlet of the tube the velocity-time profile is known very accurately. Experimentally, it was not possible to determine velocity with sufficient accuracy, thus, an approach was tried which eliminated the need for the velocity-time profile at the inlet of the tube.

It was assumed that dA/dt could be expressed as some function of $d\bar{u}/dt$, as

$$\frac{dA}{dt} = \psi\frac{d\bar{u}}{dt} \quad (26)$$

where ψ indicates some function yet to be chosen. In the early stages of attempts to numerically integrate Equations (24) and (25) before instabilities made the solution invalid, it was apparent that dA/dt was proportional to $d\bar{u}/dt$. Thus, the function relating these two quantities is the simple proportionality

$$\frac{dA}{dt} = K\frac{d\bar{u}}{dt} \quad (27)$$

where K is a constant depending on the properties of the tube and of the flowing liquid. By using Equation (27), Equations (24) and (25) can be transformed into one differential and difference equation,

$$\frac{d\bar{u}}{dt} = \frac{1}{\left(A - \frac{5}{3}\bar{u}K\right)}\left[\left(\frac{4}{3}\bar{u}^2 - \frac{C_1}{A^{1/2}}\right)\frac{\Delta A}{\Delta z} - C_2\bar{u}\right] \quad (28)$$

and when combined with Equation (27) yield a set of equations which is stable during numerical integration.

This method of solution does not require the velocity-time profile as an initial condition; however, the value of the constant, K, must be determined. To do this, it is necessary to know the volume of liquid used to create the pulse. Let this quantity be Q_p. Then

$$Q_p = \int_0^T q_p dt \quad (29)$$

and

$$q_t = q_p + q_s = \bar{u}(t)A(t) \quad (30)$$

Using Equations (29) and (30) the quantity of the pulse can be determined by

$$Q_p = \int_0^T [\bar{u}(t)A(t) - q_s]dt \quad (31)$$

To determine the correct value of K, first a value for K is assumed which allows the numerical integration of Equations (27) and (28) to be performed, obtaining an area and velocity profile at the end of the first interval along the axis of the tube. Also, the numerical integration of Equation (31) may be performed simultaneously. The calculated Q_p is then compared with the experimental value and K is adjusted accordingly. Thus a converging routine to determine the correct value of K may be incorporated into the numerical solution. When the proper K is determined, the integration is continued for the remaining intervals along the length of the tube.

EXPERIMENTAL EQUIPMENT AND PROCEDURE

The experimental equipment consisted of an expansible tube of Silastic® silicone rubber, two pressure transducers and associated equipment, and the valves and supply tanks necessary to maintain a steady flow of fluid through the tube. The pressure pulse was created by superimposing an unsteady liquid flow upon the steady flowing liquid. This pulse was introduced into the steady flowing liquid by way of a solenoid valve just prior to the entrance of the fluid into the expansible tube. The upstream pressure transducer was located near the inlet of the expansible tube while the downstream transducer was located near the center of the tube. A positive pressure of about 15 mm. Hg. was maintained in the tube by connecting the outlet of the tube to a container which had a liquid level higher than that of the tube. This was to prevent the tube from collapsing under conditions of low pressure.

Three tubes with various diameters and wall thickness and two liquids, water and ethylene glycol, were used. Several runs were made incorporating various tubes, liquids, steady state flow rate, and pulse magnitudes. From each run, two pressure-time profiles were obtained; one from each pressure transducer. These profiles were recorded by photographing the output voltage of the transducers from the screen of an oscilloscope. The pressure-time profile from the upstream transducer was then used to furnish the initial conditions for the numerical integration and the downstream pressure-time profile was used to check the results of the numerical solution.

INITIAL CONDITIONS

The initial conditions required by the numerical solution are the following:

1. The cross sectional area (or pressure)* at the upstream pressure tap as a function of time.
2. The cross sectional area (or pressure) and average axial velocity at time zero, at the end of the increments along the axis of the tube.

The first condition is the prescribed input pressure pulse while the second is the conditions which exist along the tube before the pulse enters the tube. The values of area and velocity for the second condition can be found by solving the following equation at incremental points along the tube for the condition of steady state flow.

$$Z\Big|_0^L = \left[-\frac{E\delta\pi\rho R^3}{24\mu Q} + \frac{Q}{2\pi\mu}\ln(R) + \frac{64\mu Q}{3E\delta\pi\rho R}\right]\Big|_{R_0}^{R} \quad (32)$$

This equation was developed by the authors to describe the change in radius of an expansible tube as a function of length for the conditions of steady state flow.

RESULTS

The numerical integration of Equations (27) and (28) was carried out to predict the results for two of the experimental runs. Run 39 was made at comparatively mild conditions, low viscosity and a small pressure pulse. Run 7 was indicative of the most extreme conditions tested, high viscosity and a large pressure pulse. Both runs were conducted in the most expansible tube (largest diameter

* Pressure and cross-sectional area may be interchanged by the use of the equation of state for the tube, Equation (20).

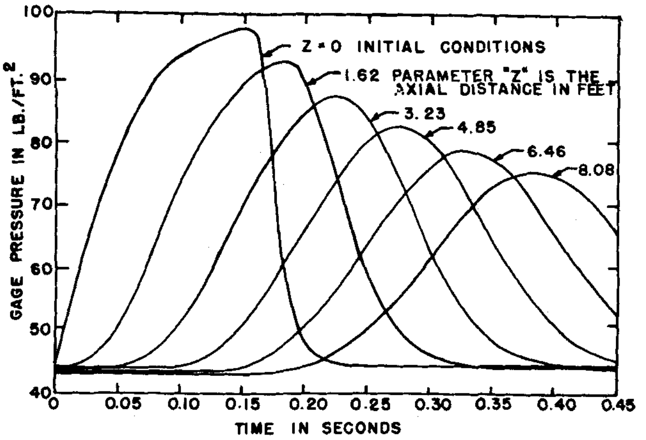

Fig. 1. Calculated pressure profiles for run 39.

and thinnest walls; I.D. = 0.0131 ft., wall thickness = 0.000696 ft.). In each case the distance between the upstream and downstream pressure transducers was 8.08 ft.

The two runs were conducted under the conditions given in Table 1.

TABLE 1.

Run No.	Steady Flow Rate, lb./sec.	Viscosity, C.P.	Density, g./cc.	Upstream Pressure, lb./sq. ft.	Pulse Quantity, ft.^{3} $\times 10^5$
7	0.01682	18.41	1.1161	154.8	75.0
39	0.00737	0.962	0.9978	44.7	8.1

Integration of Equations (27) and (29) was numerically stable and was accomplished by a computer program written in FORTRAN IV and run on an IBM 7044 computer. The numerical integration used one hundred intervals in the time domain and twenty increments along the axis of the tube. Thus twenty integrations were made along the time variable using a Runge-Kutta procedure. This yielded twenty pressure-time profiles for each run made. The results for run 39 are given in Figure 1 which shows the movement of the pressure pulse along the axis of the tube. As the pulse moves down the tube the pulse is damped by the viscous forces of the liquid. The sharp peaks of the pulse are diminished and the time duration of the pulse increases.

The accuracy of predicting the movement of the pulse and the pulse damping was excellent for this run as seen in Figure 2 which is a comparison of the experimental and calculated results at the downstream pressure tap.

Run 7 was made under the extreme conditions of high viscosity and a large pressure pulse. The pulse was so large that the tube was expanded into a range which was

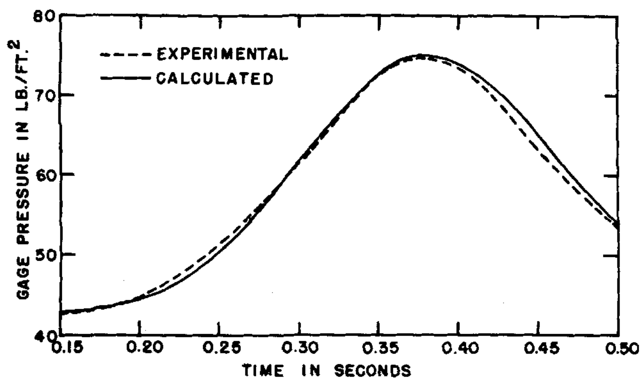

Fig. 2. Experimental and calculated results at downstream pressure tap for run 39.

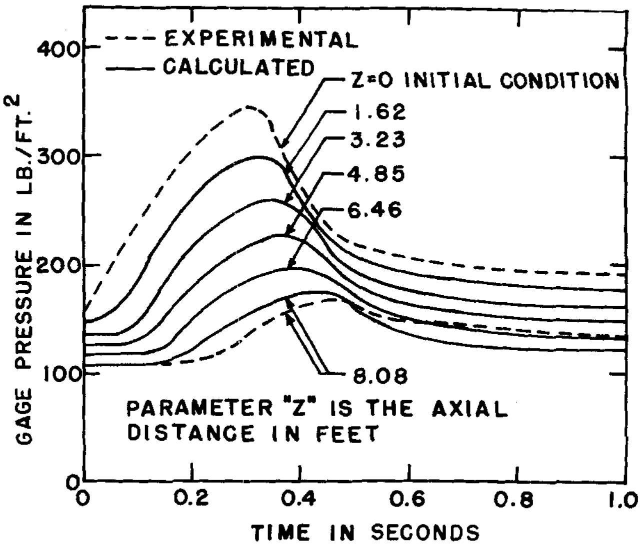

Fig. 3. Experimental and calculated profiles for run 7.

not adequately described by the equation of state of the tube. The equation of state assumes that the modulus of elasticity is a constant; for larger tube expansion, this is not true. Thus, one of the restrictions used to develop the flow model was violated. In spite of this, the predicted results were still quite good as shown in Figure 3. The prediction of the rate of movement of the pulse was not as good as for Run 39 but this was probably due to the violation of the assumption upon which the equation of state of the tube was developed. The examination of the results from these two experimental and theoretical runs indicates that the flow model developed in this article and the resulting solution is sufficient to describe accurately the movement of a pressure pulse in an expansible tube, at least for the range of variables covered in this study. The solution lends itself to the use of almost any equation of state of the tube that might be desired. This would require only a slight change in the form of the equations and can be handled quite easily numerically.

ACKNOWLEDGMENT

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NOTATION

A = cross sectional area of the tube
ΔA = $A_n - A_{n-1}$ where n indicates an increment along the axis of the tube
C_1 = $E\delta/\sqrt{\pi}/2P$
C_2 = $8\pi\nu$
E = modulus of elasticity; stress/strain
K = constant in Equation (27)
L = the distance between the upstream pressure tap and along the axis of the tube
n = the identification of some increment along the axis of the tube
P = gage pressure
$\overline{P}$ = average gage pressure defined by Equation (6)
Q = mass flow rate; mass/time
Q_p = volume of liquid used to create a pulse
q_p = volumetric flow rate of liquid causing pulse at some point along the tube
q_s = steady state volumetric flow rate in the tube
q_t = total flow rate, that of the pulsing liquid plus the

steady state flow rate
R = the inside radius of a tube at any given point
R_u = the unstressed radius of a tube
r = spatial coordinate in the radial direction
T = the duration of a pulse
t = the variable time
Δt = $t_n - t_{n-1}$
u = component of fluid velocity parallel to the axis of the tube
$\overline{u}$ = axial velocity of the fluid averaged over a cross sectional area of the tube
$\Delta \overline{u}$ = $\overline{u}_n - \overline{u}_{n-1}$
v = radial component of velocity
z = spatial coordinate in the axial direction of the tube
Δz = $z_n - z_{n-1}$

Greek Letters
δ = thickness of the tubing wall
μ = absolute viscosity
ν = kinematic viscosity
ρ = density of flowing fluid
ψ = an undefined functional relationship

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Effect of Imperfect Mixing on Autorefrigerated Reactor Stability

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This paper discusses the effects of imperfect mixing on the stability of autorefrigerated chemical reactors. Mixing is shown to strongly influence stability in these evaporatively-cooled systems. A modified van Heerden steady state stability analysis is presented. Computer simulation of a typical system illustrates the occurrence of runaways and local hot spots. Both continuous stirred tank reactors and batch reactors are considered.

Stability of perfectly mixed continuous autorefrigerated chemical reactors was discussed in a previous paper (*1*). The effects of removing the heat of reaction by evaporative cooling, instead of by conventional cooling jackets or coils, was studied by various stability analysis techniques. The decrease in latent heat of vaporization near the critical temperature was shown to be a source of positive feedback.

The purpose of this paper is to consider the case where mixing is not complete so that spacial concentration and temperature gradients can develop. This effect, resulting in local hot spots, would be expected to occur in industrial scale reactors, particularly in highly viscous or slurry systems.

Mixing is shown to significantly influence the stability of these evaporatively cooled reactors. A modified van Heerden steady state stability analysis and a computer simulation of a typical system are presented. Both continuous stirred tank reactors and batch reactors are discussed.

SYSTEM

A binary mixture of reactant A and product B is considered. Product B is the higher boiler, being of higher molecular weight and stoichiometrically requiring R_m moles of reactant A:

$$R_m A \xrightarrow{k} B \tag{1}$$

Reaction rate is assumed to be first-order in A. Vapor pressures for less volatile B and more volatile A are given by

$$\ln P_A = C_1\left(\frac{1}{T}\right) + C_2$$
$$\ln P_B = C_3\left(\frac{1}{T}\right) + C_4 \tag{2}$$

Raoult's law is used to calculate liquid and vapor compositions.

The process equipment studied is the typical reactor-condenser system shown in Figure 1. The exothermic heat

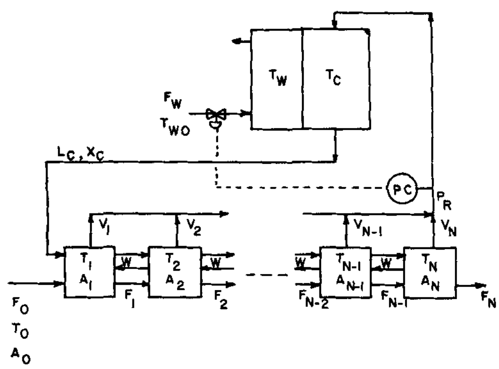

Fig. 1. Imperfectly mixed autorefrigerated reactor.